Case.	μ	Factors of p.	Discoverer.	Year.
1–5	0–4	All prime	Fermat	1640
6	5	$2^7\cdot5+1=641$ $2\cdot{}^7 52347+1=6700417$	L. Euler	1729
7	6	Unknown but composite. $2^8\cdot9\cdot7\cdot17+1=274177$ $2^8\cdot5\cdot52562829149+1$	Lucas Landry Landry and Le Lasseur	1878 1880 1880
8	7	Unknown but composite.	A. E. Western, J. C. Morehead	1905
9	8	Unknown but composite	A. E. Western, J. C. Morehead	1909
10	9	$2^{16}\cdot37+1$	A. E. Western	1903
11	11	$2^{13}\cdot3\cdot13+1$ $2^{13}\cdot7\cdot17+1$	A. Cunningham	1899
12	12	$2^{14}\cdot7+1$ $2^{16}\cdot397+1$ $2^{16}\cdot7\cdot139+1$	E. Lucas and P. Pervušin A. E. Western	1877 1903
13	18	$2^{20}\cdot13+1$	A. E. Western	1903
14	23	$2^{25}\cdot5+1$	P. Pervušin	1878
15	36	$2^{39}\cdot5+1$	Seelhoff	1886
16	38	$2^{41}\cdot3+1$	J. Cullen, A. Cunningham, A. E. and F. J. Western	1903
17	73	Unknown but composite.	J. C. Morehead	1906

The labor expended in deriving these results has been enormous. And to the layman who knows nothing of congruences in the theory of numbers, the facts found must seem almost to border on the miraculous. For, even when $\mu = 10$, a case not yet solved, p contains 309 digits; but when $\mu = 36$, p is a number of more than twenty trillion digits. Concerning it Lucas remarked[1] "la bande de papier qui le contiendrait ferait le tour de la Terre." For $\mu = 73$, Ball states that the digits in p "are so numerous that, if the number were printed in full with the type and number of pages used in this book [*Mathematical Recreations*, fifth edition, 1911, 508 pages], many more volumes would be required than are contained in all the public libraries of the world."

In not less than seven places[2] did Fermat refer to $2^{2^\mu} + 1$ as an expression for

10, 12 (Western), 13, 16. *Proc. Lond. Math. Soc.* (2), Vol. 1, 1903, p. 175; abstract of paper read May 14, 1903.

11. A. CUNNINGHAM, *Brit. Assoc. Rept.*, 1899, pp. 653–4.

12, 14. E. LUCAS, *Atti Accad. Torino*, Vol. 13 (1877–8), p. 271 [27 Jan., 1878]. *Mélanges math. astr. acad. Petersb.*, Vol. 5 (1874–81), p. 505, 519 or *Bull. Acad. Pétersb.* (3) vol. 24, 1878, col. 559; (3) Vol. 25, 1879, col. 63; communication by V. Buniakovskij of results, for $\mu = 12$ and 23, found by J. Pervušin, in Nov. 1877 and Feb. 1878.

15. P. SEELHOFF, *Zeitschrift math. u. Phys.*, Vol. 31, 1886, p. 380.

17. J. C. MOREHEAD, *Bull. Amer. Math. Soc.*, Vol. 12, 1906, pp. 449–451.

[1] E. LUCAS, *Théorie des nombres*, Vol. 1, Paris, 1891, p. 51.

[2] Letter dated August [?] 1640 to Frenicle (*Oeuvres de Fermat*, Vol. 2, 1894, p. 206; letter dated 18 October, 1640, to Frenicle (*Oeuvres*, Vol. 2, 1894, p. 208; *Varia Opera*, Toulouse, 1679, p. 162; Brassine's *Précis*, Toulouse, 1853, pp. 142–3); letter dated 25 December, 1640, to Mersenne (*Oeuvres*, Vol. 2, pp. 212–213); "De solutione problematum geometriconum per curvas simplicissimas et unicuique problematum generi proprie convenientes, Dissertatio tripartita" (*Oeuvres de Fermat*, Vol. 1, 1891, pp. 130–131; French translation, Vol. 3, 1896, p. 120; *Varia Opera*, 1679 [reprint, 1861], p. 115); letter dated 29 August, 1654, to Pascal (*Œuvres de Pascal*,

determining a series of prime numbers. The earliest reference is in a letter of August[?], 1640, to Frenicle. He wrote:

"But here is something which pleases me greatly: it is that I am almost persuaded that numbers of the progression $2^0, 2^1, 2^2, 2^3, \cdots$, augmented by 1, are prime numbers, as

3, 5, 17, 257, 65 537, 4 294 967 297,

and the following of 20 digits

18 446 744 073 709 551 617; etc.

I have not an exact demonstration, but I have excluded such a large number of divisors by infallible proofs, and have so many side lights which bear out my thought, that I would find difficulty in convincing myself of error."

In October of the same year Fermat again wrote to Frenicle

"I have not yet demonstrated the exclusion of all divisors in that beautiful proposition which I sent you and which you verified for me with respect to the numbers, 3, 5, 17, 257, 65537, etc For, although I can prove the exclusion of most divisors and show the probability of exclusion for the rest, I am not yet able to demonstrate the necessary truth of the proposition, concerning which however, I have no more doubt at this moment than I had previously. If you have a sure proof you will oblige me by communicating the same to me; for, after that, nothing can keep me back in these matters."

Fourteen years later Fermat had to write to Pascal,

"The demonstration of the proposition is very difficult and I confess to you that I have not yet fully found it; I should not propose that you seek it, had I already reached the goal."

In a somewhat similar vein he demanded a demonstration of Digby in 1658. Indeed, at five different times, last in 1658, Fermat (died 1665) carefully noted that he lacked a rigorous proof that $2^{2^\mu} + 1$ is always a prime number. It is remarkable that he overlooked the fact that this was not prime for $\mu = 5$, since he himself made a remark on the possible factors of numbers of the form $2^m \pm 1$, from which it may be shown that the prime factors of $2^{32} - 1$ must be primes of the form $64n + 1$.[1] From this, Euler's factors 641 and 6700417 could be deduced at once. Fermat's exact statement here, as elsewhere, tends but to confirm the belief in the accuracy of what he wrote with regard to his celebrated theorem (the proof of which has baffled the greatest mathematicians ever since): "I have found for this a truly wonderful proof."[2]

Gauss's Statement of his Polygon Results. In two passages the implication to be drawn from what Klein has written is, that Gauss published a proof that a regular polygon of p sides can not be constructed by ruler and compasses if p is a prime not of the form $2^k + 1$. The passages to which I refer are:[3]

Vol. 4, Paris, 1819, p. 384; *Oeuvres de Fermat*, Vol. 2, 1894, pp. 309–310); letter to Sir Kenelm Digby, sent by Digby to Wallis, 19 June, 1658 (*Oeuvres de Fermat*, Vol. 2, 1894, pp. 402, 404–5; French translation of the Latin, Vol. 3, 1896, p. 314, 316); letter dated August, 1659 to Carcavi, copy sent by Carcavi to Huygens 14 August, 1659 (*Corresp.* de Huygens no. 651; *Oeuvres de Fermat*, Vol. 2, pp. 433–434).

[1] W. W. R. BALL, *Math. Recreations and Essays*, fifth ed., London, 1911, p. 40, is authority for this last statement. I have a distinct impression that it has also been made by some first-class authority, but I have vainly searched Fermat's works for its verification.

[2] *Oeuvres de Fermat*, Vol. 1, 1891, p. 291: "Cujus rei demonstrationem mirabilem sane detexi."

[3] English edition, pp. 2, 16; German edition, pp. 2, 13; French edition, pp. 10, 26.

(1) "Gauss added other cases [to Euclid's] by showing the *possibility* of the division into parts where p is a prime number of the form $p = 2^{2^\mu} + 1$, and the *impossibility* for all other numbers" (the italics here and in (2) are mine); (2) "Gauss extended this series of numbers [Euclid's] by showing that the division is *possible* for every prime number of the form $p = 2^{2^\mu} + 1$ but *impossible* for all other prime numbers and their powers." Now the implication referred to above is not correct, as Professor Pierpont interestingly set forth in his paper "On an undemonstrated theorem of the *Disquisitiones Arithmeticae*."[1] That is, Gauss *did not give a proof* of the "impossibility" referred to in the quotations. But after proving the "possibility" as described above he uses the following words:

"As often as $p-1$ contains other prime factors besides 2, we arrive at higher equations,[2] namely, to one or more cubic equations, if 3 enters once or oftener as a factor of $p-1$, to equations of 5th degree if $p-1$ is divisible by 5, etc. And we can prove with all rigour that these equations cannot be avoided or made to depend upon equations of lower degree; and although the limits of this work do not permit us to give the demonstration here, we still thought it necessary to signal this fact in order that one should not seek to construct other polygons than those given by our theory, as, for example, polygons of 7, 11, 13, 19 sides, and so employ one's time in vain."

To carry Gauss's reasoning further, by supplementing what I have given in the last section, it will be of interest to follow Professor Pierpont. He says:

"Having laid down the theory for polygons of a prime number of sides, Gauss now turns his attention to polygons of any number of sides, $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_\nu^{a_\nu}$ where $p_1, p_2, \cdots$, are the prime factors of n. These he disposes of in a very summary fashion by declaring, without any attempt at proof, that they can be constructed then and only then when

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_\nu}\right)$$

contains no other factor than 2."

"That this is a sufficient condition follows at once from an easy extension of Gauss's method as developed when n is a prime. It is, however, vastly more important to know that *only* these polygons can be geometrically constructed as thereby the theory of regular polygons, as far as their construction by ruler and compasses is concerned, is made complete. That is, in a given case we can decide whether the polygon is constructible, and in case that it is, Gauss's theory gives us the necessary directions to construct it."

In the first part of his paper Professor Pierpont shows "that the condition which Gauss gave as necessary is in fact such."

Geometrical Constructions of the Regular 17-side. The remark of Klein[3] that we possess as yet no method of construction of the regular polygon of seventeen sides, based upon considerations purely geometrical, is a little curious, since several constructions of this kind have been given. One by Erchinger was indeed reported by Gauss in 1825.[4] The construction is as follows:

[1] *Bull. Amer. Math. Soc.*, Dec., 1895, Vol. 2, pp. 77–83. *Cf.* also L. E. Dickson in *Monographs on Topics of Modern Mathematics*, London and N. Y., 1911, p. 386.

[2] In his earlier discussion of an inscribed polygon of p sides, Gauss considers the equation $x^p - 1 = 0$ and the resulting equation got by dividing out the factor $x - 1$ where p = prime.

[3] English edition, pp. 24, 32; German edition, pp. 19, 26; French edition, pp. 35, 43.

[4] *Göttingische gelehrte Anzeigen*, Dec. 19, 1825, no. 203, p. 2025; *Werke*, Vol. 2, pp. 186–7. To Art. 365 of the *Disquisitiones Arithmeticae* Gauss added this note in his handwriting: "Circulum in 17 partes divisibilem esse geometrice, deteximus 1796 Mart. 30." *Cf. Werke*, Vol. 1, p. 476.

Let D, B, G, A, I, F, C, E be points on a line determined by constructions about to be given. Let AB be a line of any length. Produce it both ways to

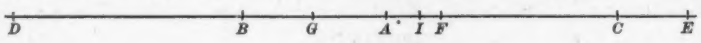

Fig. 1.

C and D so that,

$$AC \times BC = AB \times BD = 4\overline{AB}^2.$$

Further determine the points E, G, on both sides of CA produced so that,

$$AE \times EC = AG \times CG = \overline{AB}^2;$$

and find the point F on the side A of the line BA produced, such that

$$AF \times DF = \overline{AB}^2.$$

Finally divide AE in I so that

$$AI \times EI = AB \times AF,$$

where AI is the smaller, and EI the larger part of AE. Then construct a triangle, in which each of two sides equals AB, the third being equal to AI. About this triangle describe a circle; then AI will be one side of the regular inscribed polygon of seventeen sides.

Gauss particularly remarks that the author gave a purely synthetic proof of this construction.

Another synthetic construction and proof dated "Dublin, 17th October, 1819" was published by Samuel James in the *Transactions of the Irish Academy*.[1] The following earlier solution by John Lowry was Prize Question 410 in *The Mathematival Repository* for 1819:[2] "Draw the radius CO at right angles to the

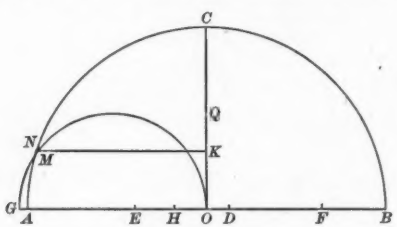

Fig. 2.

diameter AB;[3] on OC and OB, take OQ equal to the half, and OD equal to an eighth part of the radius; make DE and DF each equal to DQ, and EG and FH respectively equal to EQ and FQ; take OK a mean proportional between OH and OQ, and through K draw KM parallel to AB, meeting the semicircle described on OG in M; draw MN parallel to OC cutting the given circle in N, the arc AN is the seventeenth part of the whole circumference."

[1] Vol. 13 (1818), pp. 175–187; paper read Jan. 24, 1820.
[2] New Series, Vol. 4, p. 160. Lowry's proof occupies pp. 160–168.
[3] O is the middle point of AB.

Irrationality of π.—Klein wrote:[1] "After 1770 critical rigour gradually began to resume its rightful place. In this year appeared the work of Lambert: *Vorläufige Kentnisse für die so die, Quadratur . . . des Cirkuls suchen.* Among other matters the irrationality of π is discussed. In 1794 Legendre in his *Éléments de Géométrie* showed conclusively that π and π^2 are irrational numbers." The implication of this note is that Lambert did not discuss the irrationality of π conclusively and that Legendre did. How both of these points of view are essentially incorrect will appear in what follows. Klein was simply reproducing the erroneous statements of Rudio;[2] but after Pringsheim's careful study in 1898,[3] Lambert's proof emerged as "auserordentlich scharfsinnig und im wesentlichen vollkommen einwandfrei," while Legendre's remained "in Bezug auf Strenge hinter Lambert weit zurück."

As in the later proof of the transcendence of π, so here when its irrationality was in question, discussion of e is fundamental. The irrationality of e and e^2 was substantially shown by Euler in 1737[4] and he gave the expression for e as a continued fraction on which Lambert's proofs of the irrationality of e^x, $\tan x$ and π rest. Starting with Euler's development[5]

$$\frac{e-1}{2} = \frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\,\frac{1}{18+}\ \text{etc.},$$

Lambert found

$$\frac{e^x-1}{e^x+1} = \frac{1}{2/x+}\,\frac{1}{6/x+}\,\frac{1}{10/x+}\,\frac{1}{14/x+}\ \text{etc.},$$

and since

$$\frac{e^x-1}{e^x+1} = \frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \tanh\frac{x}{2} = \frac{1}{i}\tan\frac{ix}{2}, \quad \text{if} \quad z = \frac{ix}{2},$$

$$\tan z = \frac{1}{1/z-}\,\frac{1}{3/z-}\,\frac{1}{5/z-}\,\frac{1}{7/z-}\,\frac{1}{9/z-}\ \cdots.$$

He then proved the theorems:

1. *If x is a rational number different from zero, e^x can never be rational.*

For $x = 1$, we have as special case the irrationality of e.

2. *If z is a rational number different from zero, $\tan z$ can never be rational.*

For $z = \pi/4$, $\tan \pi/4 = 1$, and hence as a special case the irrationality of π.

[1] English edition, p. 59; German edition, p. 46; French edition, p. 72.

[2] F. RUDIO: *Archimedes, Huygens, Lambert, Legendre, vier Abhandlungen über die Kreismessung*. Leipzig, 1892, p. 56f. This error is also reproduced by B. Calò in ENRIQUES'S *Fragen der Elementargeometrie*, II. Teil, 1907, p. 315; by D. E. Smith in YOUNG'S *Monographs on Topics of Modern Mathematics*, 1911, p. 401. The matter was correctly set forth by T. VAHLEN in *Konstruktionen und Approximationen*, Leipzig, 1911, pp. 319ff.

[3] A. PRINGSHEIM: "Ueber die ersten Beweise der Irrationalität von e und π." *Sitzungsberichte der mathematisch-physikalischen Classe der k. b. Akademie der Wissenschaften zu München*, Bd. 28, 1899, pp. 325–337.

[4] "De fractionibus continuis," *Comment. acad. de Petrop*, Vol. 9, 1744, p. 108. Presented to St. Petersburg Academy, March, 1737.

[5] L. EULER: *Introductio in analysin infinitorum*. Tomus Primus, Lausannae, 1748, p. 319. This work was finished in 1745; *Cf.* G. Eneström, *Verzeichnis* etc., Erste Lieferung, p. 25.

The part of Lambert's "Vorläufige Kentnisse" to which Klein refers contains some formulae without proof, and no analytical developments, and was rather intended to serve as a popular survey of the treatment of the topic. With it must be considered the scientifically remarkable "Mémoire" of 1767.[1] Here "mit minutiöser Genaugkeit" Lambert proves the convergence of the expression for $\tan z$ as a continued fraction. Pringsheim dwells on the "astounding" nature of these considerations at this period in the history of mathematical thought. For of such considerations Legendre was innocent, as well as the great Gauss in his 1812 memoir on hypergeometric series, and others, till a much later period.

"Thus the Lambert memoir contains the *first*, and for many years, the *only* example of what we now consider really rigorous developments of functions as converging continued fractions, in particular, that for $\tan z$ given above."

Constructions in General with Ruler and Compasses. While the Greeks built elaborate structures with propositions involving Euclidean methods, Descartes was the first to discuss the question, What geometrical constructions are, and what are not, theoretically possible by these methods? This discussion harks back to the famous, anonymously printed, geometry[2] of 1637, in which the first book was entitled: "The problems which can be constructed by employing circles and straight lines only."[3] Before referring to Klein in this connection, I wish to give some indications of Descartes's argument.[4]

Descartes's Discussion. The book begins with the remark that all the problems of geometry can be reduced to a form in which the only condition of their construction is the determination of the lengths of certain right lines; and that all that is necessary to the determination of the length of a line from sufficient data is the power of interpreting geometrically the five arithmetical operations of addition, subtraction, multiplication, division, and the extraction of roots in such a manner that, the quantities operated on being lengths, the result shall also be a length. This is to be effected by the introduction of the unit of length (if necessary) as a factor or divisor, so as to reduce the construction to finding a fourth proportional to three given lines, or a mean proportional, or several mean proportionals, between two given lines. The actual constructions for multiplication, division, and the extraction of the square root are then given; roots of higher order being reserved for subsequent consideration. Descartes thus disposes of a difficulty which had undoubtedly been felt in the geometric interpretation of expressions above the third degree. But he remarks that any expression which

[1] "Memoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques." Lu en 1767. Printed in 1768 in *Hist. de l'acad. royale des sciences et belles-lettres*, Berlin, Année 1761 (!), pp. 265–322.

[2] Discours *de la methode pour bien conduire sa raison, et chercher la verite dans les sciences. Plus La Dioptrique, Les Meteores et La Geometrie, qui sont des essais de cette Methode.* A Leyde, 1637; in *Oeuvres* de Descartes publiés par Charles Adam et Paul Tannery, Tome VI, Paris, 1902, pp. 1–418.

[3] Adam-Tannery edition, *l. c.*, pp. 369–387.

[4] I follow Professor J. M. Peirce closely in part of his article, "References in Analytic Geometry," *Harvard University Library Bulletin*, Nos. 8, 10, 11, 1878–1879, pp. 157–158, 246–250, 289–290.

denotes a line must be homogeneous and of the first degree, if its value is independent of the choice of the unit. He next speaks of the formation and reduction of equations, in the solution of a problem, and observes that those problems are *plane*, that is, can be solved by straight lines and circles on one flat surface, when the final equation, containing one unknown quantity, is of a degree not higher than the second. The reason of this, of course, is that two circles, or a right line and a circle, intersect in only two points. The actual constructions for the different forms of a quadratic equation are then given. It is important to note that Descartes pays no attention to *negative solutions*. Thus he considers only the three forms,[1] $x^2 = +ax + b^2$, $x^2 = -ax + b^2$, $x^2 = +ax - b^2$, disregarding altogether the form $x^2 = -ax - b^2$, and he constructs only *one* root for each of the first two equations, while he constructs two for the third.[2] Indeed, Descartes seems to have reached a less advanced point on this subject than had already been attained by an earlier writer. Albert Girard, a Dutchman, in his *Invention Nouvelle en l'Algèbre* (Amsterdam, 1629) lays down the true principle of negative quantities, exactly and broadly.

After giving the four forms of construction of a root of a quadratic equation, Descartes makes the important remark that *all the problems of ordinary geometry can be constructed by what is contained in these four figures*. And he continues: "I do not believe that this fact was noted by the ancients; for otherwise they had not taken the trouble to write so many large works where the order alone of their propositions tells us that they did not have the true method to discover them all, but that they have simply collected those which they have met."

As a decisive test of the power of his geometry, Descartes next takes up a problem which Pappus of Alexandria cites at the beginning of the 7th book of his *Mathematical Collections*, stating that neither Euclid nor Apollonius nor any other had been able to give a general solution of it. This problem may be stated, in effect, as follows: "Given in any plane n right lines, and also either n or $(n - 1)$ other right lines, to find the locus of a point such that the product of its distances from the first set of lines measured on lines making any given angles with them, shall be in a given ratio to the product of the distances of the point from the second set of lines measured on lines making given angles with them." With the discussion of this problem, the first book of Descartes's Geometry closes.

Klein's Discussion. In the "Introduction" to Klein's book he announces the fundamental problem stated in the question at the beginning of this section. He

[1] Descartes does not use the sign =, but a sign which is perhaps an abbreviation of "aequatur."

[2] In fact, there is no evidence that Descartes perceived that the negative root of an equation (the *false* roots as he calls them) could have any real meaning in a geometric construction; and it is a familiar observation that they generally give a solution of the problem from which they spring, of a kind not contemplated at the outset, and inadmissible if the problem is rigidly interpreted. Later, indeed, in giving constructions for equations of the third and fourth degrees, he exhibits the false roots as well as the true; but in these cases, either of the two sorts of lines may be true roots, according to the sign of one of the coefficients in the equation. Moreover, though Descartes was undoubtedly aware that opposite signs always corresponded to opposite directions, and repeatedly speaks of the possible variations that his constructions thus admit, he does not seem to have apprehended this fact as a *principle*, which might be laid down, once for all, at the outset of the discussion; and he never uses a letter which denotes a quantity as admitting a negative value.

then remarks: "To define sharply the meaning of the word 'construction,' we must designate the instruments which we propose to use in each case. We shall consider," he continues, "(1) straight edge and compasses, (2) compasses alone, (3) straight edge alone, (4) other instruments used in connection with straight edge and compasses."

After two very compact paragraphs, Klein states the following fundamental theorem: "*The necessary and sufficient condition that an analytic expression can be constructed with a straight edge and compasses is that it can be derived from the known quantities by a finite number of rational operations and square roots.*" When this is considered with the following theorem of Chapter I, we have a criterion for identifying the problems under consideration: "*If x, the quantity to be constructed, depends only upon rational expressions and square roots, it is a root of an irreducible equation $\phi(x) = 0$, whose degree is always a power of 2.*" Whence it is shown that if this degree of an irreducible equation is not a power of 2, it cannot be solved by square roots.

Other References. Now the implications in the discussion leading to these theorems are very numerous and no student has mastered all the principles involved until he has approached the subject from other points of view. Foremost to be recommended is Castelnuovo's article adapted from his Projective Geometry[1] for the second part of *Fragen der Elementargeometrie*[2] by Enriques. The articles by Daniele[3] and Giacomini[4] in this same volume should also be carefully studied. Among other results here indicated we find:

That every problem which can be solved with ruler and compasses can also be solved, (*a*) with compasses alone; (*b*) with ruler alone, if we are given a fixed circle with its center in the plane of construction; (*c*) with a ruler alone whose edges are parallel; (*d*) with a ruler alone, whose edges are not parallel, but converge to a point. Many problems can also be solved (*e*) with a ruler and segment-carrier. Theorem (*a*) was first enunciated and proved by Lorenzo Mascheroni (1750–1800) in his famous book *La Geometria del compasso*, Pavia, 1797.[5] An elegant proof by Adler based upon the method of inversion was given in 1890.[6] Mascheroni constructions are treated in English, by Cayley in a paper of 1885,[7] and by Hobson in his recent Presidential Address[8] before the Mathematical Association.

[1] *Lezioni di geometria analitica e proiettiva*, Roma-Milano, 1905. The article is entitled, "Uber die Lösbarkeit der geometrischen Aufgaben mit den elementaren Instrumenten; Betrachtungen vom Standpunkte der analytischen Geometrie." This and the other articles have also just appeared, in slightly modified form, in *Questioni riguardanti le mathematiche elementari*, raccolte e coordinate da F. Enriques, Volume II, Bologna, 1914.

[2] Leipzig, 1907.

[3] "Uber die Lösung der geometrischen Aufgaben mit dem Zirkel."

[4] "Uber die Lösung der geometrischen Aufgaben mit dem Lineal und den linealen Instrumenten: Betrachtungen vom Standpunkte der projektiven Geometrie."

[5] French edition by Carette, Paris, 1798; second edition, 1828. A German translation of the first French edition, "vermehrt mit der Theorie vom Gebrauche des Proportional-Zirkels und mit einer Sammlung zur Uebung von mehr denn 400 rein geom. Sätzen, von J. P. Grüson" was published at Berlin in 1825.

[6] A. Adler, "Zur Theorie der Mascheronischen Konstruktionen," *Situngsberichte der Wiener Akademie*, Bd. 99, Abt. IIa, 1890, p. 910ff.

[7] *Messenger of Mathematics*, vol. 14, 1885, pp. 179–181; *Collected Papers*, Vol. 12, pp. 314–317.

[8] "On geometrical constructions by means of the compass," *Mathematical Gazette*, Vol. 7, March, 1913, pp. 49–54.

To Poncelet (1788–1867) Theorem (*b*) is due,[1] although the result is frequently referred to Steiner who published, in 1833, the little classic, *Die geometrischen Konstruktionen, ausgeführt mittelst der geraden Linie und eines festen Kreises.*[2] But many of the theorems, including the fundamental one, here found, had been already given by Poncelet and Lambert. By omitting to state that the middle point of the given circle must be known[3] Steiner inaccurately formulates the fundamental theorem. In a recent course of lectures Hilbert proved that knowledge of the position of the middle point was essential and suggested as a problem; "How many given circles in a plane are necessary in order to determine with ruler alone, the center of one of them?" In 1912 D. Cauer showed: (1) *If two circles intersect in imaginary points in the finite part of the plane, it is impossible to determine the middle point of either circle with ruler alone;* (2) *A center may be determined if the circles cut in real points, touch, or are concentric.*[3] About the same time J. Grossmann discovered[4] a result which leads us to Theorem

(*f*): Every problem which can be solved with ruler and compasses can also be solved with ruler alone if we are given 3 linearly independent circles, no two concentric, in the plane of construction.

Theorems (*c*) and (*d*) were proved by Adler in 1890.[5] The former is however really implied in Steiner's work referred to above. In the *Foundations of Geometry*[6] by Hilbert, a certain Theorem (*e*) plays a special role and the ideas thus suggested were elaborated by Feldblum in his dissertation.[7] While the theorem and its early applications are due to Hilbert, it is remarkable to find that Lambert introduced the compasses in exactly the same way as the segment-carrier was employed in later times: "Geometrical constructions are all based on the ruler and compasses. . . . The ruler is used simply for drawing straight lines, *and the compasses serve the purpose of marking off lengths on them and acting as carrier, as well as of drawing angles and giving to lines their proper position.*"[8]

To Plato (429–348 B.C.) has been attributed the chief influence in determining that among the instruments which might have been chosen for developing a system of geometry, the ruler and compasses were selected. Already in the tenth century we find the Arabian, Aboûl Wafâ of Bagdad, somewhat limiting, apparently, these means at his disposal by solving problems with ruler as before but with a compass whose arms were open at a constant angle.[9] With such instru-

[1] *Traité des propriétés projectives des figures*, Paris, 1822, pp. 187–190.

[2] An abridgment in French was published by A. Lévy in *Nouv. Annales de Math.*, 1908, pp. 390–409.

[3] Page 68.

[4] *Mathematische Annalen*, Vol. 73, 1912, pp. 90–94. See also Vol. 74, 1913, pp. 462–464.

[5] A. ADLER, "Uber die zur Ausführung geometrischer Konstruktionen notwendigen Hilfsmittel," *Sitzungsberichte der Wiener Akademie*, Bd. 99, 1890, Abt. IIa, p. 846 ff.

[6] Fourth German edition, Leipzig, 1913; English edition by E. J. Townsend, Chicago, 1902.

[7] M. FELDBLUM, *Ueber Elementar-Geometrische Constructionen.* Diss. Gottingen, Warschau, 1899.

[8] J. H. LAMBERT, *Beiträge zum Gebrauche der Mathematik und deren Anwendung*, Berlin, 1765, Teil I, pp. 23–24. The latter part of the last sentence quoted has not been translated very literally; the original runs: "und der Zirkel dient, um sie zu fassen und abzutragen, desgleichen auch um Winkel zu ziehen, und den Linien ihre behörige Lage zu geben."

[9] *Woepecke*, "Analyse et extrait d'un recueil de constructions géométriques par Aboûl Wafâ," *Journal Asiatique*, 1855.

ments in the sixteenth century, Cardano, Ferraro, and Tartaglia solved certain problems in Euclid. Without reference to the published solutions of Ferraro and Cardano, Giovanni Battista Benedetti published at Venice in 1553 a treatise entitled: *Resolutio omnium Euclidis problematum, aliorumque ad hoc necessario inventorum, una tantummodo circini data apertura.* For a detailed history of this phase of geometry the reader may consult the monograph by Kutta.[1] In English, I recall but two titles in this connection; the first is a rare pamphlet translated from the Dutch by Joseph Moxon in 1677,[2] the second is an article by the late Dr. J. S. Mackay.[3]

To our series we may now add Theorem

(g): Every problem which can be solved with ruler and compasses, can also be solved with a ruler and one fixed aperture of the compasses.

In concluding this section, it should be remarked with regard to the quotation made above from Klein's Introduction, that his "consideration" of constructions with compasses alone, straight edge alone, and other instruments used in connection with straight edge or compasses, is practically confined to fugitive historical notes on pages 33–34, 47 (English edition).

Concluding Remarks. In the latter part of Chapter IV Klein has made a slight slip (English ed., p. 77; Ger. ed., p. 63; Fr. ed., p. 92). In connection with the equation $y = e^x$, he wrote: "To an algebraic value of x corresponds a transcendental value of y, and conversely." "Conversely" leads us to the statement, to a transcendental value of y corresponds an algebraic value of x. But proof of this has nowhere been given; indeed the result is not true, in general. To correct, delete "conversely" and add: "To an algebraic value of y corresponds a transcendental value of x."

It is to be hoped that the editors of a new edition will be moved

(1) to add a proof that the only regular polygons which it is possible to construct with ruler and compasses are those the number P of whose sides can be expressed in the form

$$2^{\alpha} \cdot (2^{2^{\alpha_1}} + 1) \cdot (2^{2^{\alpha_2}} + 1) \cdot (2^{2^{\alpha_3}} + 1) \cdots (2^{2^{\alpha_s}} + 1)$$

where $\alpha_1 \cdots \alpha_s$ are distinct positive integers and each $2^{2^{\alpha_i}} + 1$ is a prime; α and *one* of the series $\alpha_1, \alpha_2, \cdots \alpha_s$ may be zero.

(2) To at least indicate the connection with continued fractions, and the actual method of determining a and b of the theorem stated on page 17, "if μ and ν are prime to each other, we can always find integers a and b positive or negative, such that $1 = a\mu + b\nu$."

(3) To add the proof that

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[1] W. M. KUTTA, "Zur Geschichte der Geometrie mit constanter Zirkelöffnung," *Nova Acta Abh. der Kaiserl. Leop.-Carol. Deutschen Akademie der Naturforscher*, Vol. 71, Halle, 1897, pp. 71–101.

[2] *Compendium Euclidis Curiosi: or, geometrical Operations. Shewing how with a single opening of the compasses and a straight ruler all the propositions of Euclid's first five books are performed.* London, 1677. Moxon does not tell us who the author of the original Dutch treatise was.

[3] "Solutions of Euclid's Problems, with a rule and one fixed aperture of the compasses, by the Italian geometers of the sixteenth century," *Proc. Edinb. Math. Soc.*, Vol. 5, 1887, pp. 2–22.

$$(x+1)^{p(p-1)} + (x+1)^{p(p-2)} + \cdots + (x+1)^p + 1 = 0$$

will take the form $\qquad x^{p(p-1)} + p \cdot \chi(x),$

if p is a prime number, and if "$\chi(x)$ is a polynomial with integral coefficients whose constant term is 1" (pages 22–23).

(4) To point out that although the *Conchoid of Nicomedes* is used in the text to trisect an angle, this application of the curve was the discovery of Pappus (about 300 A.D.), and *not* of Nicomedes (about 180 B.C.).[1] Nicomedes used the curve for the duplication of the cube.[2]

(5) To add the indication of proof that a prime number greater than any number we please *exists*. This is needed in the proof of the transcendence of e.

(6) To add, page 61, the proof that $\mathrm{Lim}_{n=\infty} x^n/\lfloor n = 0$.

(7) To make the following corrections:

Page 12, *for* $F(x) = C\cdot[\phi(x)]^r$ *read* $F(x) = C_1\cdot[\phi(x)]^r$.

Page 14, line 8 should read

$$+ C_1 \sum_{\nu=1}^{\nu=N} \sum_{r} e^{|k_\nu|} c_r k_\nu^r q_{r,k_\nu} + C_2 \sum_{\nu=1}^{\nu=N'} e^{|l_\nu|} c'_r l_\nu^r q_{rl_\nu} + \cdots = 0.$$

Page 34, *for* "with the straight edge and one fixed circle we can solve every quadratic equation," *read* "with the straight edge and one fixed circle, the center being given, we can solve every quadratic equation for which line-segments corresponding to the coefficients are given."

Page 72, line 7, *for* b^{Np} *read* $b^{N'p}$; *for* l_N *read* $l_{N'}$.

Also in the theorem on page 5, some ambiguity would be avoided by setting $\phi(x) = 0$ for $f(x) = 0$.

Note. Since this article was written I have seen a new portion, Bd. III, Heft 5, of the *Encyklopädie der mathematischen Wissenschaften*, published at Leipzig on June 8, 1914. It contains a section by J. Sommer on "Elementare Geometrie vom Standpunkte der neueren Analysis aus," pages 773–858, and about half of the section, pages 859–962, by M. Zacharias on "Elementargeometrie und elementare nicht-Euklidische Geometrie in synthetischer Behandlung." Many parts of this Heft will be of interest in connection with questions discussed above. As to the Gaussian polygons, reference might have been given to paragraph 22 of L. O. Hölder's section of the *Encyklopädie*, published in 1899 and entitled, "Galois'sche Theorie mit Anwendungen."

ON THE TRISECTION OF AN ANGLE AND THE CONSTRUCTION OF REGULAR POLYGONS OF 7 AND 9 SIDES.

By L. E. DICKSON, University of Chicago.

1. Purpose and Plan of this Note. Frequently a wide-awake student who has learned how to bisect any angle asks if every angle can be trisected and, if not, why not; after learning how to construct regular polygons of 3, 4, 5, 6, 8

[1] Pappus, ed. Hultsch, p. 246.

[2] *Cf.* CANTOR, *Vorlesungen über Geschichte der Math.*, Bd. I, 3 Aufl., 1907, p. 351.

and 10 sides, he is apt to ask about the missing ones of 7 and 9 sides. Having several times received a first aid call from teachers of these inquisitive students, the writer would find it convenient to be able to refer to an exposition of these questions which is as elementary as possible. It is the purpose of this note to present such a treatment.

Moreover, it seems necessary that these questions be discussed publicly at regular intervals in order to keep down the number of angle-trisectors, who are partly unable and largely unwilling to understand the standard proofs of the impossibility of these constructions by means of ruler and compasses, but prefer to attempt to make the issue depend upon their own alleged construction involving always a confusing mass of lines and circles and always a child-like error.

With either class of readers, the use of imaginary numbers is not convincing. Hence they are not employed in this note, even though the imaginary roots of unity enter naturally into the questions concerning regular polygons. Moreover, the entire discussion is not beyond a college freshman.

2. The Cubic Equations. In the problem of the duplication of a cube, we take as the unit of length a side of the given cube, and seek the length x of a side of another cube whose volume is double that of the given cube; thus

$$x^3 = 2. \tag{1}$$

In the problem of the trisection of a given angle A, we are given a line of length $\cos A$ and seek a line of length $\cos (A/3)$. For, if we lay off the unit of length AB on one arm of angle A and draw the perpendicular BC to the other arm, the number of units of length in AC is $\cos A$ or $-\cos A$, according as A is an acute or obtuse angle. We employ the well-known trigonometric identity

$$\cos A = 4 \cos^3 \frac{A}{3} - 3 \cos \frac{A}{3}.$$

Multiply each member by 2 and set $x = 2 \cos (A/3)$. Thus

$$x^3 - 3x = 2 \cos A.$$

We are to prove that an arbitrary[1] angle A cannot be trisected by ruler and compasses. It suffices to prove this for the angle $A = 120°$. Then $\cos A = -1/2$ and the cubic is

$$x^3 - 3x + 1 = 0. \tag{2}$$

After we have proved that angle 120° cannot be so trisected and hence that angle 40° cannot be constructed by ruler and compasses, it will follow that a regular polygon of nine sides cannot be so constructed, since the angle at the center subtended by one side is $\frac{1}{9}\, 360° = 40°$.

Finally, the problem of the construction of a regular polygon of seven sides by ruler and compasses is equivalent to the construction of angle B containing

[1] Certain angles, like $A = 180°$, can be trisected. Since $\cos 60° = \frac{1}{2}$, the cubic then has the root $x = 1$. Hence this case does not invalidate our general theorem in § 4.

$\frac{360}{7}$ degrees and hence to the construction of a line of length $x = 2 \cos B$. We have

$$\cos 3B = \cos (360° - 3B) = \cos (7B - 3B) = \cos 4B,$$

$$\cos 3B = 4 \cos^3 B - 3 \cos B,$$

$$\cos 4B = 2 \cos^2 2B - 1 = 2 (2 \cos^2 B - 1)^2 - 1.$$

After multiplication by 2 and setting $x = 2 \cos B$, we get

$$x^3 - 3x = 4(\tfrac{1}{2}x^2 - 1)^2 - 2,$$

$$0 = x^4 - 4x^2 + 2 - x^3 + 3x = (x - 2)(x^3 + x^2 - 2x - 1).$$

But $x = 2$ would give $\cos B = 1$, whereas B is acute. Hence[1]

$$x^3 + x^2 - 2x - 1 = 0 \tag{3}$$

3. Our Cubic Equations Have No Rational Roots. Suppose for example that equation (2) has the root a/b, where a and b are integers with no common (integral) divisor greater than unity. Then

$$\frac{a^3}{b^3} - 3\frac{a}{b} + 1 = 0, \qquad \frac{a^3}{b} = 3ab - b^2 = \text{integer}.$$

Thus if $b \neq \pm 1$, b has a divisor greater than unity in common with a, contrary to hypothesis. Hence $b = \pm 1$ and the root is an integer.

If a root x of (2) is an integer, it divides x^3 and $3x$ and hence also the constant term 1, so that $x = \pm 1$. By trial, neither $+1$ nor -1 is a root. Hence (2) has no rational root.

The same discussion applies step by step to equation (3). In the case of (1), we must try also the divisors ± 2.

Hence each of our problems has led us to a cubic equation with rational coefficients having no rational root. Each problem is therefore impossible in view of the next theorem.

4. General Theorem. *It is not possible to construct by ruler and compasses a line whose length is a root or the negative of a root of a cubic equation with rational coefficients having no rational root.*

We begin by investigating the nature of a positive number p such that a line of length p can be constructed by ruler and compasses. The ends of this line as well as other points found in the course of the construction are located as the intersections of straight lines and circles. Consider the equations of these lines and circles referred to a fixed pair of rectangular axes, the y-axis not being parallel to any of our straight lines. The equation of any one of our lines is

$$y = mx + b. \tag{1}$$

[1] We can derive this equation and the corresponding ones for other regular polygons without the use of trigonometry, making use only of a theorem on chords in a circle. *Cf.* Dickson, *Annals of Mathematics*, 1894, p. 73.

Another line intersecting this has an equation

$$y = m'x + b'$$

and the coördinates of their point of intersection,

$$x = \frac{b' - b}{m - m'}, \; y = \frac{mb' - m'b}{m - m'},$$

are rational functions of the coefficients of the lines.

To find the coördinates of the intersections of (1) with the circle

$$(x - e)^2 + (y - f)^2 = r^2,$$

we eliminate y and obtain a quadratic equation for x. Thus x, and hence also y, involves no irrationality (besides irrationalities already appearing in m, b, e, f, r) other than a square root.

Finally, the intersections of two circles are given by the intersections of one of them with their common chord, so that this case reduces to the preceding.

Hence the coördinates of the various points located by the construction, and therefore also the length p of the segment joining two of them, are found by a finite number of rational operations and extractions of real square roots, performed upon rational numbers or numbers obtained by such operations. By way of example, note that the side of a regular pentagon inscribed in a circle of radius unity is

$$\tfrac{1}{2}\sqrt{10 - 2\sqrt{5}}.$$

This point settled, consider a cubic equation with rational coefficients and having a constructible root r. Either r is rational or else it involves a real square root. In the latter case, we obtain a second root of the cubic by changing the sign of this square root in the expression for r. Then the third root of the cubic must be rational, since otherwise there would be, as before, a pair of roots in addition to the first pair. Hence in every case the cubic has a rational root, so that the denial of the general theorem stated at the beginning of this section leads to a contradiction. We have merely outlined in a rough way the final step of the proof. The argument in detail is accessible in books by Klein[1] and the writer;[2] it is based upon a systematic classification of the square roots involved in r, but employs only elementary algebraic principles.

The final step in the proof can be made in a few lines by means of the Galois theory of equations, which is based upon the theory of groups.

For a more elaborate elementary discussion of these special problems and the general problem relating to regular polygons, the reader may consult the eighth article in *Monographs on Modern Mathematics*, Longmans, Green and Co., 1911, where further references are given on page 386.

[1] *Elementarmathematik vom höheren Standpunkte aus*, Leipzig, 1908, vol. 1, p. 125, and 2d ed., 1911.

[2] *Elementary Theory of Equations*, Wiley and Sons, 1914, p. 90.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

The Madison Colloquium Lectures on Mathematics. BY LEONARD EUGENE DICKSON and WILLIAM FOGG OSGOOD. Published by the American Mathematical Society, 501 West 116th St., New York, 1914. 230 pages.

The seventh colloquium of the American Mathematical Society was held in connection with the twentieth summer meeting at the University of Wisconsin during the week of September 8–13, 1913. Professor W. F. Osgood, of Harvard University, gave five lectures on "Topics in the theory of functions of several complex variables," and Professor L. E. Dickson, of the University of Chicago, gave five lectures on "Invariants and the theory of numbers." These lectures have been published by the society as Volume IV of its series of colloquium lectures. The price of the book is $2.00; to members of the society, $1.50. The first three volumes of the series are as follows: I, The Boston Colloquium, 1903. II, The New Haven Colloquium, 1906. III, The Princeton Colloquium, 1909. Before the society became a national organization, a colloquium was held at Evanston, Illinois, in 1893. Professor Klein of Göttingen was the sole speaker. The first edition of this colloquium was exhausted, and a second edition was published by the society under the title: The Evanston Colloquium Lectures on Mathematics. The society held a colloquium at Buffalo in 1896, one at Cambridge in 1898, and one at Ithaca in 1901. The society did not publish the lectures delivered at these colloquia.

W. H. BUSSEY.

Archimedes' Werke. Mit modernen Bezeichnungen herausgegeben und mit einer Einleitung versehen von SIR THOMAS L. HEATH. Deutsch von DR. FRITZ KLIEM. O. Häring, Berlin, 1914.

No greater tribute could be paid to the excellence of the English translation of the works of Archimedes, as made by Heath, than the fact that a German scholar should take this edition as the basis for a retranslation into German. This, too, is the more surprising in view of Heiberg's new and revised Greek and Latin edition of Archimedes' works which is in course of publication, two of the three projected volumes having already appeared. Heiberg's text makes full use of the many additions to our knowledge of Archimedes which have been made in the thirty-odd years since the publication by him of the critical text with Latin translation; seventeen years have elapsed since Heath's work appeared.

Fortunately the German edition has received the careful attention both of Heath and Kliem so that the additions of recent times are incorporated. Thus the newly discovered work of Archimedes, *The Method of Archimedes, concerning theorems (proved by) mechanics—to Eratosthenes,* is incorporated in this volume, based upon the translation published by Heath in 1912 as a supplement to his earlier work. This treatise was found by Heiberg in a manuscript of the tenth century, a palimpsest, now in the Metochion at Constantinople but formerly

from the monastery of the Holy Grave at Jerusalem. This treatise should be familiar to every teacher of elementary mathematics for it gives us clear insight into the methods of work of the greatest mathematician of antiquity. The theorems of the *Method* are entirely within the grasp of any who have studied analytic geometry. The quadrature of a segment of a parabola and the determination of the volume and surface of a sphere are included in the discussion. Besides the translation published by Heath, an English translation is available in the *Monist* for 1909,[1] a German translation in the *Bibliotheca Mathematica*, Vol. 7, third series, pp. 321–363, 1907, and a French description in the *Revue générale des Sciences*, 1907, as well as the more critical Greek edition for philologists in *Hermes*, 1907, Vol. 42, pp. 235–297. How easily an important work may be lost without any trace of it in literature is evident from this treatise which was formerly known only through a chance reference by Suidas to a commentary on it by Theodosius, and the citation by Heron of three of the theorems (and this latter reference first appeared in print in 1903).

Fragments of the *Stomachion* by Archimedes are also preserved in the Constantinople papyrus. Dr. Kliem has added to his valuable work not only these brief passages on this ancient puzzle but also the more complete information preserved in an Arabic translation and published by Suter in Vol. IX of the *Abhandlungen zur Geschichte der Mathematik*. In this geometrical puzzle some fourteen ivory pieces which fit together to make a square are to be arranged in different ways so as to represent a ship, a sword, and other objects. The name of Archimedes has long been associated with this game but only recently has the definite evidence been obtained showing that the great genius of the Greek world did not disdain mathematical puzzles.

This work can be heartily recommended to all students of mathematics, for Dr. Kliem has made available in one volume the works of Archimedes which have come down to us. The book is also a fine specimen of the printer's art. Of corrections necessary I have noted only on page 429, line 14, reference to "Satz 8," instead of "Satz 9."

LOUIS C. KARPINSKI.

Elementary Theory of Equations. By L. E. DICKSON. John Wiley and Sons, New York, 1914. v+184 pages. $1.75.

This book occupies a middle ground in difficulty, being too advanced for the average freshman, but still of an elementary character, suitable for a second course in the theory of equations. It is such a book as may be read with profit by any one who wants an *exact* statement and rigorous proof of the elementary theorems —not involving group-theory or invariants—concerning algebraic equations; a work of value to all teachers of algebra, whether elementary or advanced. In particular every teacher of algebra should read the proof of the fundamental theorem of algebra and the work on graphing; while every teacher of geometry, should read the proofs given in Chap. VIII relating to the trisection of an angle

[1] Republished in pamphlet form by the Open Court Publishing Company.

duplication of the cube, and construction of regular polygons of 7, 9, and 17 sides. An exact treatment of these topics cannot but be of aid to anyone interested in elementary mathematics.

The early introduction of graphing and the use of derivatives in finding " bend points " enable the writer at the beginning of the book to give a discussion of the discriminant of the cubic $x^3 - 3lx + q = 0$, while the work throughout the book is rendered clear by the use of graphs. In the first chapter we are given also the graphical solution of a quadratic. Complex numbers are next introduced in a soul-satisfying way. This chapter should be read by everyone who thinks that complex numbers are " imaginary " and that we gain nothing by their use except to make certain equations have roots.

" The fundamental theorem of algebra " is also treated in a satisfactory way, the graphical proof being clear and elementary. However it tacitly assumes that if a polynomial in x and y is positive at a point A and negative at a second point B, it will vanish at some point between A and B on any curve joining them—a theorem which might have been explicitly stated, particularly since the corresponding theorem for one variable is carefully given (page 13). Also (page 52, line 13) why the word " perhaps "? " A region or perhaps regions " would convey the meaning better. We must have one such region just as assuredly as a region when Y is positive. In fact the statements about these regions are far from clear and are not proved or carefully explained. But when a proof of this important theorem is given which rests on such elementary principles of algebra and graphics as this one does, surely even minor criticism is ill-timed.

The theorem that an integral root of an equation with integral coefficients divides the constant term might well be supplemented by the similar theorem that if an equation with integral coefficients has a fractional root α/β, α must divide the constant term and β the coefficient of the highest power of x. This gives in general a simpler way to find such roots than that given on page 62.

The treatment of symmetric functions is unusually complete and careful.

The reviewer is glad to see in an accessible place a treatment given to the problems of trisecting an angle and duplicating a cube. These puzzle students and often teachers, partly because the problem is not clearly understood, and partly because there is so obviously a solution; and yet their impossibility may readily be made plausible to a student familiar with coördinate geometry and is here rigorously proved in an elementary way. We are also shown why there can be no construction, in the Euclidean sense, of regular polygons of 7 and 9 sides.

The usual theorems for the isolation of the roots are given in Chap. IX, as well as some theorems that are not found in most textbooks. The use of elementary calculus allows a clear treatment and a complete solution of the problem, " given an equation to locate its real roots," while the methods of Chap. X show how to compute them. Besides Horner's well-known method for the numerical computation of roots, Newton's is given and emphasized as one that is effective for non-algebraic as well as for algebraic equations; and Gräffe's little known but very ingenious scheme of solution by forming equations whose roots are powers

of the roots of the given equation, and Lagrange's solution by continued fractions are also explained.

In Chap. XI determinants receive a clear natural treatment, while the subject of resultants and discriminants is carefully and rigorously discussed in the closing chapter.

On the whole in this book there is much to praise and little with which to find fault.

ELIJAH SWIFT.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

SPECIAL NOTICE. In proposing problems and in preparing solutions, contributors will please follow the form established by the MONTHLY, as indicated on the following pages.

In particular, a solution should be preceded by the number of the problem, the name and address of the proposer, the statement of the problem, and the name and address of the solver.

The solution should then be given with careful attention to legibility, accuracy, brevity without obscurity, paragraphing and spacing, having in mind the form in which it will appear on the printed page.

Please use paper of letter size, write on one side only, leaving ample margins, put one solution only on a single sheet and include only such matter as is intended for publication.

Drawings must be made *clearly* and *accurately* and an extra copy furnished on a *separate sheet* ready for the engraver.

Unless these directions are observed by contributors, solutions must be entirely rewritten by the committee or else rejected.

Selections for this department are made two months in advance of publication.

Please send all solutions direct to the chairman of the committee.

MANAGING EDITOR.

ALGEBRA.

When this issue was made up solutions of 410–19 had been received. Solutions of 406 and 420 are desired.

420. Proposed by ELBERT H. CLARKE, Purdue University.

Given the infinite series,

$$\frac{a}{r}+\frac{b}{r^2}+\frac{a+b}{r^3}+\frac{a+2b}{r^4}+\frac{2a+3b}{r^5}+\cdots,$$

in which a and b are any numbers and where each numerator after the first two is the sum of the two preceding numerators. To find the region of convergence and the sum of the series.

This problem is a generalization of one solved in the January number of the MONTHLY.

421. Proposed by C. N. SCHMALL, New York City.

Give a trigonometrical solution of the general quadratic equation.

GEOMETRY.

When this issue was made up solutions of 437–38–39–40–43–45–47–48–54 had been received. Solutions of 427–30–32–33–44–46 are desired.

449. Proposed by H. E. TREFETHEN, Colby College.

Find a tetrahedron with the face angles at one vertex in arithmetical progression and its six edges expressed in integers.

450. Proposed by W. L. WATSON, Moundsville, W. Va.

If three straight lines AA', BB', CC', drawn from the vertices of a triangle ABC to the opposite sides, pass through a common point O within the triangle, then

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1.$$

CALCULUS.

When this issue was made up solutions of 354–5–6–7–8–9, 361–2–6–9 had been received. Solutions of 332–39–40–42–64–65 are desired.

370. Proposed by PAUL CAPRON, United States Naval Academy.

The surface of a right circular cone, having the semi-vertical angle α, is cut by two planes which intersect the axis at the same point, one at right angles to the axis, the other making the angle $(90° - \beta)$ with the axis. Show that, if the lateral surface of the right cone is S_1 and that of the oblique cone S_2,

$$S_2 = \sum_1^\infty T_n, \text{ where } T_1 = S_1, \qquad T_{n+1} = T_n \times \frac{2n+1}{2n} (\tan \alpha \tan \beta)^2.$$

371. Proposed by B. F. FINKEL, Drury College.

Prove that the shortest distance between two curves or two surfaces is normal to both.

MECHANICS.

When this issue was made up solutions of 271–4–5, 288–9, 292–3–4–5–7 had been received. Solutions of 268–9, 277–8–9, 286–7 are desired.

298. Proposed by C. N. SCHMALL, New York City.

A person desires to throw a stone so as to strike the greatest possible blow at a point in a smooth vertical wall at a height h above the ground. If his strength is sufficient to throw the stone vertically upwards to a height $2h$, show that he must stand at a distance $2h$ from the wall. (The resistance of the air, and the height of the hand are not taken into account.)

299. Proposed by B. F. FINKEL, Drury College.

A cone rests in two fluids which do not mix, with its vertex downwards and its base in the surface of the upper fluid; to find how much its density must be increased, that it may rest with its base in the common surface of the fluids.

[From Walton's *Hydrostatical Problems*.]

NUMBER THEORY.

When this issue was made up solutions of 207–10–12–13–16–18–20 had been received. Solutions of 189, 191–4–6, 200–5–8–9–11–13–14–15 are desired.

222. Proposed by A. H. HOLMES, Brunswick, Maine.

Find rational values for m and n in order that $(m^2 + 1)^2/m^2 + (n^2 - 1)^2/n^2$ may be the square of an integer.

223. Proposed by THOS. E. MASON, Bloomington, Indiana.

Show that

$$\frac{(rst)!}{t!(s!)^t (r!)^{st}}$$

is an integer (r, s, t being positive integers). Generalize to the case of n integers r, s, t, u, ···. (Carmichael, *Theory of Numbers*, p. 28.)

SOLUTIONS OF PROBLEMS.

ALGEBRA.

408. Proposed by EMMA M. GIBSON, Drury College, Mo.

Show that if n is a positive integer, the sum of the series

$$1 - \frac{2n-1}{\lfloor 1} + \frac{(2n-1)(2n-2)}{\lfloor 2} - \cdots + (-1)^{n-1}\frac{(2n-1)(2n-2)\cdots(n+1)}{\lfloor n-1}$$

is

$$\frac{(-1)^{n-1}(2n-2)(2n-3)\cdots(n+1)n}{\lfloor n-1}.$$

SOLUTION BY HOWARD C. FEEMSTER, York College, Neb.

Let

$$S_1 = 1, \qquad S_2 = -\frac{2n-2}{\lfloor 1}, \qquad S_3 = \frac{(2n-2)(2n-3)}{\lfloor 2},$$

$$S_4 = -\frac{(2n-2)(2n-3)(2n-4)}{\lfloor 3}.$$

Assume

$$S_r = \frac{(-1)^{r-1}(2n-2)(2n-3)\cdots(2n-r)}{\lfloor r-1},$$

then

$$S_{r+1} = \frac{(-1)^{r-1}(2n-2)(2n-3)\cdots(2n-r)}{\lfloor r-1} + \frac{(-1)^{r}(2n-1)(2n-2)(2n-3)\cdots(2n-r)}{\lfloor r}$$

$$= \frac{(-1)^{r}(2n-2)(2n-3)(2n-4)\cdots(2n-r-1)}{\lfloor r},$$

which is of the same form as S_r. Hence,

$$S_n = \frac{(-1)^{n-1}(2n-2)(2n-3)(2n-4)\cdots[2n-(n-1)](2n-n)}{\lfloor n-1}$$

$$= \frac{(-1)^{n-1}(2n-2)(2n-3)(2n-4)\cdots(n+1)n}{\lfloor n-1},$$

as required.

Note. This is simply half of the expansion, $(1-1)^{2n-1}$, the other half, S_n', is equal to this but opposite in sign. $S_n + S_n' = 0$.

Also solved by HORACE OLSON, A. M. HARDING, and GEO. W. HARTWELL.

409. Proposed by C. E. GITHENS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelopiped so that its diagonal shall be rational.

SOLUTION BY WALTER C. EELLS, United States Naval Academy.

Let x, y, z be the three edges, a the diagonal, and w the hypotenuse of the right triangle of which x and y are sides. Then we seek solutions in integers of the equation $x^2 + y^2 + z^2 = a^2$.

The following method is a generalization of the well known formulas for rational right triangles; namely, if the legs be denoted by $M^2 - N^2$, and $2MN$, and the hypotenuse, therefore, by $M^2 + N^2$, then all integral values of M and N, $(M \neq N)$, will give rational integral sides.

Let $x = 2MN$, $y = M^2 - N^2$, then $w = M^2 + N^2$. This hypotenuse may be thought of under two forms:

$A.$ $\quad M^2 + N^2 = m^2 - n^2$, then $z = 2mn$, $\quad a = m^2 + n^2$;

$B.$ $\quad M^2 + N^2 = 2m'n'$, $\quad$ then $z = m'^2 - n'^2$, $a = m'^2 + n'^2$.

Under either form it will be necessary to resolve $M^2 + N^2$ into factors, p, q, or $2m'n'$, in all possible ways.

$A.$ $\mathbf{M^2 + N^2 = m^2 - n^2 = (m + n)(m - n) = p \cdot q}$, in which p and q are integers and $p \neq q$. For convenience take $p > q$. Then $m = (p + q)/2$ and $n = (p - q)/2$, whence p and q must be both of form (1) $2k + 1$, or both of form (2) $2k'$.

(1) *p and q both odd.* Then $M^2 + N^2$ must be odd, and M and N one even and one odd. There will be two cases to consider, according as $M^2 + N^2$ is (a) prime or (b) composite.

(a) $M^2 + N^2$ prime. The only factors are itself and unity. For a given N and M there is only one set of values p, q.

Example: $N = 1$, $M = 2$. $x = 2MN = 4$. $y = M^2 - N^2 = 3$. $M^2 + N^2 = 5$. $p = 5$, $q = 1$. $m = (p + q)/2 = 3$. $n = (p - q)/2 = 2$. $z = 2mn = 12$. $a = m^2 + n^2 = 13$. This gives the smallest rational parallelopiped, $(x, y, z, a) = (4, 3, 12, 13)$.

(b) $M^2 + N^2$ composite. There are two or more sets of values of p, q.

Example: $N = 1$, $M = 8$, $x = 16$, $y = 63$, $M^2 + N^2 = 65$,

$$\begin{cases} p = 65, \ q = 1; \ m = 33, \ n = 32; \ z = 2112, \ a = 2113; \\ p' = 13, \ q' = 5; \ m' = 9, \ n' = 4; \ z' = 72, \ a' = 97. \end{cases}$$

This gives the two solutions: (16, 63, 2112, 2113), (16, 63, 72, 97).

Remark. It is desirable to so restrict the methods given that only relatively prime values of x, y, z, a, shall appear. Proofs for the statements given to attain this object will not be given, but they are not difficult to supply, based on simple algebra and number theory considerations.

When N and M have a common factor of the form $2k + 1$, $M^2 + N^2$ is composite [(1), (b) above], and one of the set of values p, q will have the same common factor $2k + 1$; and $(2k + 1)^2$ will be a common factor of x, y, z, a. This may be divided out giving a reduced solution with x', y', z', a' relatively prime, which will appear elsewhere in the series of relatively prime values found,

e. g., $N = 3$, $M = 6$, gives the solution (36, 27, 108, 117) by the method above, for $p = 15$, $q = 3$. Dividing by 9 there results the solution (4, 3, 12, 13) already given in (1), (a).

But *all* combinations where N and M have a common factor $2k + 1$ should

not be excluded,—only those for which in addition p and q have that factor. Thus, for $N = 3$, $M = 6$, as just given, there may be two prime solutions for $(p, q) \equiv (45, 1)$ or $(9, 5)$. They are (36, 27, 1012, 1013) and (36, 27, 26, 53).

Note. In actually computing a, it is easier not to use $a = m^2 + n^2$, but $a = 2mn + q^2$,

$$(a = m^2 + n^2 = 2mn + m^2 + n^2 - 2mn = 2mn + (m - n)^2 = 2mn + q^2),$$

since m and n need not be squared, $2mn$ is already computed, and q is usually small.

(2) *p and q both even.* Then $M^2 + N^2 = 4k$ and M and N are both even, and so have the common factor 2. It is easily shown that x, y, z, a have the common factor 2, and accordingly *no* prime solutions will be given in this case. It can be shown that the reduced forms of all these solutions appear elsewhere in prime form directly,

(a) By method A, (1) if N or M is of form $4k$, and $M^2 + N^2$ prime,

(b) One solution by A, (1) and others by B (below) if N or M is of form $4k$, and $M^2 + N^2$ composite,

(c) By method B if neither N nor M is of form $4k$.

B. $\mathbf{M^2 + N^2 = 2m'n'}$. For convenience the primes are dropped. Then $x = 2MN$, $y = M^2 - N^2$, $z = m^2 - n^2$, $a = m^2 + n^2$ where m and n $(m > n)$ are all possible factors of $(M^2 + N^2)/2$. Since $M^2 + N^2$ must be even, M and N must be (1) both odd or (2) both even.

(1) *N and M even.*

Example: $N = 2$, $M = 4$, $x = 16$, $y = 12$,

$$\begin{cases} m = 10, & n = 1, & z = 99, & a = 101, \\ m' = 5, & n' = 2, & z' = 21, & a' = 29. \end{cases}$$

Two solutions are (16, 12, 99, 101), (16, 12, 21, 29).

(2) *N and M odd.*

Example: $N = 1$, $M = 3$, $x = 6$, $y = 8$, $z = 24$, $a = 26$.

The example in (2) gives a solution having the common factor 2. It is easy to show that this is always the case when N and M are both odd. To find relatively prime solutions by method B, there should be excluded the following cases. (Details of proof omitted.)

(a) $N = 2k + 1$, $M = 2k' + 1$.

(b) $N = 2k$, $M = 2k'$, for solutions only where m and n are both even. For again it develops that x, y, z, a have the common factor 2,

e. g., $N = 4$, $M = 8$. Exclude for $(m, n) \equiv (20, 2)$ or $(10, 4)$ but retain for $(m, n) \equiv (40, 1)$.

(c) When N and M have a common factor $2k + 1$, for solutions only where m and n have the same common factor $2k + 1$,

e. g., $N = 6$, $M = 12$. Exclude solutions where $(m, n) \equiv (30, 3)$ and $(15, 6)$, but retain them where $(m, n) \equiv (90, 1)$, $(45, 2)$, $(18, 5)$, $(10, 9)$ to secure prime solutions only.

GENERAL REMARKS.

1. The methods of A and B above suffice to determine an indefinite number

of prime solutions of the required equation. Those for which a is less than 100, with the method, A or B, by which they are obtained, are as follows:

x, y, z, a		x, y, z, a	
(4, 3, 12, 13)	A	(12, 5, 84, 85)	A
(16, 12, 21, 29)	B	(64, 48, 39, 89)	B
(24, 32, 9, 41)	B	(16, 63, 72, 97)	A
(36, 27, 28, 53)	A	(56, 33, 72, 97)	A

2. For M and N less than 15, there are twelve groups of two solutions each for the same diagonal a,

$$\text{e. g.,} \begin{cases} 16,\ 63,\ 72,\ 97\ A \\ 56,\ 33,\ 72,\ 97\ A \end{cases} \qquad \begin{cases} 96,\ 28,\ 75,\ 125\ B \\ 108,\ 45,\ 44,\ 125\ A \end{cases}$$

$$\begin{cases} 140,\ 171,\ 24420,\ 24421\ A \\ 220,\ 21,\ 24420,\ 24421\ A \end{cases} \qquad \begin{cases} 40,\ 96,\ 153,\ 185\ B \\ 72,\ 135,\ 104,\ 185\ A \end{cases}$$

3. If x and y are fixed values, in certain cases there are two or more parallelopipeds possible with rational diagonals, according to the number of ways $M^2 + N^2$ may be resolved into factors when composite.

Examples:

(1) $N = 1$, $M = 8$, (16, 63, 2112, 2113),
(16, 63, 72, 97).

(2) $N = 6$, $M = 12$, (144, 108, 19, 181),
(144, 108, 299, 349),
(144, 108, 2021, 2029),
(144, 108, 8099, 8101),

and if composite solutions be allowed the two additional ones

(144, 108, 189, 261),
(144, 108, 891, 909).

4. The method may be further extended to find the diagonal of the four dimensional figure (or n dimensional figure), analogous to the parallelopiped. E. g., to find solutions in integers of $x^2 + y^2 + z^2 + v^2 = a^2$. One such is $(x, y, z, v, a) \equiv (3, 4, 12, 84, 85)$. An easier one is $(a, a, a, a, 2a)$, when a is any integer.

5. For all values of M and N less than 15, there are (unless some error in computation has been made) 125 prime rational parallelopipeds. The smallest is given above. The largest is (420, 29, 88620, 88621).

The following table gives these 125 solutions, arranged according to size of a, and indicating the method by which each was derived. There are 78 solutions found by method A, and 47 by method B.

x	y	z	a		x	y	z	a	
4	3	12	13	A	48	55	2664	2665	A
16	12	21	29	B	364	27	2652	2677	A
24	32	9	41	B	40	96	2703	2705	B
36	27	28	53	A	112	180	2805	2813	B
12	5	84	85	A					
64	48	39	89	B	80	84	3363	3365	B
16	63	72	97	A	36	77	3612	3613	A
56	33	72	97	A	84	13	3612	3613	A
					240	44	3717	3725	B

x	y	z	a	
16	12	99	101	B
96	28	75	125	B
108	45	44	125	A
8	15	144	145	A
36	77	132	157	A
84	13	132	157	A
48	20	165	173	B
144	108	19	181	B
40	96	153	185	B
72	135	104	185	A
140	171	60	229	A
220	21	60	229	A
192	80	105	233	B
224	132	69	269	B
96	128	231	281	B
32	60	285	293	B
120	64	273	305	B
24	7	312	313	A
44	117	300	325	A
144	108	299	349	B
216	63	272	353	A
300	125	228	397	A
24	32	399	401	B
20	21	420	421	A
56	192	375	425	B
24	143	408	433	A
180	189	380	461	A
96	28	621	629	B
56	192	609	641	B
48	20	675	677	B
12	35	684	685	A
104	153	672	697	A
224	132	651	701	B
40	9	840	841	A
80	84	837	845	B
84	187	828	853	A
156	133	828	853	A
360	81	800	881	A
36	27	1012	1013	A
216	63	1000	1025	A
32	60	1155	1157	B
336	52	1131	1181	B
196	147	1188	1213	A
48	140	1365	1373	B
280	96	1353	1385	B
28	45	1404	1405	A
264	23	1392	1417	A
80	39	3960	3961	A

x	y	z	a	
224	132	4221	4229	B
120	64	4623	4625	B
72	65	4704	4705	A
20	99	5100	5101	A
48	140	5475	5477	B
60	91	5940	5941	A
112	15	6384	6385	A
96	128	6399	6401	B
160	36	6723	6725	B
108	45	6844	6845	A
336	52	7221	7229	B
44	117	7812	7813	A
100	75	7812	7813	A
144	108	8099	8101	B
88	105	9384	9385	A
56	192	9999	10001	B
24	143	10512	10513	A
144	17	10512	10513	A
192	80	10815	10817	B
140	51	11100	11101	A
112	180	11235	11237	B
72	135	11704	11705	A
132	85	12324	12325	A
120	119	14280	14281	A
240	44	14883	14885	B
52	165	14964	14965	A
180	19	16380	16381	A
224	132	16899	16901	B
104	153	17112	17113	A
176	57	17112	17113	A
168	95	18624	18625	A
28	195	19404	19405	A
84	187	21012	21013	A
156	133	21012	21013	A
280	96	21903	21905	B
140	171	24420	24421	A
220	21	24420	24421	A
216	63	25112	25113	A
60	221	26220	26221	A
208	105	27144	27145	A
336	52	28899	28901	B
120	209	29040	29041	A
196	147	30012	30013	A
180	189	34060	34061	A
264	23	35112	35113	A
260	69	36180	36181	A
252	115	38364	38365	A
240	161	41760	41761	A

x	y	z	a		x	y	z	a	
64	48	1599	1601	B	312	25	48984	48985	A
160	36	1677	1685	B					
60	11	1860	1861	A	308	75	50244	50245	A
					300	125	52812	52813	A
144	108	2021	2029	B					
16	63	2112	2113	A	364	27	66612	66613	A
56	33	2112	2113	A	360	81	68080	68081	A
96	28	2499	2501	B					
					420	29	88620	88621	A

Also solved in less detail by E. E. WHITFORD, C. E. FLANAGAN, and G. I. HOPKINS.

GEOMETRY.

Note. The following remark should be made in connection with the solution of Geometry **417** in the September issue:

Three planes determine *either* a point (which may be at infinity if the intersections of the planes in pairs are parallel) *or* a straight line in which the planes are concurrent (which may be at infinity if the planes are parallel). Some of the 220 points required may therefore prove to be replaced by such lines of concurrency of three planes. Whenever the word *line* is used in this solution it refers to a straight line in which three planes are concurrent.

CALCULUS.

337. Proposed by R. P. BAKER, University of Iowa.

Show that for a, b relatively prime integers,

$$\int_0^1 |\cos 2\pi ax + \cos 2\pi bx|\, dx = \frac{2}{\pi ab}\left\{\frac{a+b}{\sin\frac{\pi}{a+b}} - \frac{a-b}{\sin\frac{\pi}{a-b}}\right\}$$

or

$$= \frac{1}{\pi ab}\left\{(a+b)\cot\frac{\pi}{2(a+b)} - (a-b)\cot\frac{\pi}{2(a-b)}\right\}$$

according as a and b are both odd or one of them is even.

SOLUTION BY THE PROPOSER.

Take the first case with $a > b$.

The zeros of the integrand in the path of integration are

$$\xi_{1r} = \frac{2r-1}{2(a+b)}, \qquad r = 1, 3, 5, \cdots (a+b)$$

and

$$\xi_{2s} = \frac{2s-1}{2(a-b)}, \qquad s = 1, 3, 5, \cdots (a-b).$$

If these are $\xi_1, \xi_2, \xi_3, \cdots \xi_{2a}$ in order of magnitude the integral is

$$\frac{1}{\pi ab}\left[b\sin 2\pi a\xi + a\sin 2\pi b\xi\right]_{\xi_2,\ \xi_4,\ \cdots\ \xi_{2a}}^{\xi_1,\ \xi_3,\ \cdots\ \xi_{2a-1}}.$$

Consider first the sines of $2\pi b\xi_{1r}$. The rth is preceded by $r-1$ of its own set and by s of the other set where

$$s = \left[\frac{a-b}{2(a+b)}(2r-1) + \tfrac{1}{2}\right]; \quad [\alpha] \equiv \text{greatest integer in } \alpha.$$

The expression is an integer only when $\xi_{2s} = \frac{1}{2}$ which does not occur in this case.

For the integral we have

$$\sin\pi\left\{\frac{b(2r-1)}{2(a+b)} + (r-1) + \left[\frac{a-b}{2(a+b)}(2r-1) + \tfrac{1}{2}\right]\right\} = \sin\pi(L + [M]) \text{ say.}$$

Now $L + M = 2r - 1$ and M is not an integer. Hence $(L + [M]) = 2r - 2$ and the sine in question enters the integral positively. The set ξ_{1r} contains all odd multiples of $(a+b)/2$ and b is prime to $(a+b)$ and odd. The angles are then all odd multiples of $\pi/(a+b)$ and all being distinct and in the first two quadrants must be precisely the set

$$\pi k/(a+b), \quad k = 1, 3, 5, \cdots (a+b) - 1.$$

The sum of this set of sines contributes to the integral

$$\frac{2}{\pi ab}\left(\frac{a}{\sin\pi/(a+b)}\right).$$

A similar count for the set $\sin 2\pi a\xi_{1r}$, using $L' - [M]$ instead of $L + [M]$, adds a term of the same form, a and b being interchanged.

Similarly the roots of the ξ_{2s} set with a contribute four times the sum of sines of all odd multiples less than the $(a-b)$th of $\pi/(a-b)$. With b the same set occurs with negative sign.

A similar method gives the formula in case 2.

350. Proposed by R. P. BAKER, University of Iowa.

Find a general formula for d^ny/dx^n in terms of d^ky/dt^k and d^kx/dt^k.

SOLUTION BY J. W. CLAWSON Collegeville, Pa.

Let δx, δy be increments of x, y when t takes the increment δt. Then, by Taylor's Theorem,

$$\delta y = \delta x\frac{dy}{dx} + \frac{\overline{\delta x}^2}{\underline{2}}\frac{d^2y}{dx^2} + \frac{\overline{\delta x}^3}{\underline{3}}\frac{d^3y}{dx^3} + \cdots, \tag{1}$$

$$\delta y = \delta\frac{dy}{dt} + \frac{\overline{\delta t}^2}{\underline{2}}\frac{d^2y}{dt^2} + \frac{\overline{\delta t}^3}{\underline{3}}\frac{d^3y}{dt^3} + \cdots, \tag{2}$$

$$\delta t = \delta x\frac{dt}{dx} + \frac{\overline{\delta x}^2}{\underline{2}}\frac{d^2t}{dx^2} + \frac{\overline{\delta x}^3}{\underline{3}}\frac{d^3t}{dx^3} + \cdots. \tag{3}$$

From (2) and (3) we get

$$\delta y = \left(\delta x\frac{dt}{dx} + \frac{\overline{\delta x}^2}{\underline{2}}\frac{d^2t}{dx^2} + \cdots\right)\frac{dy}{dt} + \left(\delta x\frac{dt}{dx} + \frac{\overline{\delta x}^2}{\underline{2}}\frac{d^2t}{dx^2} + \cdots\right)^2\frac{1}{\underline{2}}\frac{d^2y}{dt^2} + \cdots. \tag{4}$$

Comparing (1) and (4) and equating coefficients of same powers of δx we get

$$\frac{1}{\lfloor n}\frac{d^n y}{dx^n} = \sum_{k=1}^{k=n} C_k \frac{1}{\lfloor k}\frac{d^k y}{dt^k},$$

where C_k is the coefficient of $\overline{\delta x}^n$ in the expansion of

$$\left[\delta x \frac{dt}{dx} + \frac{\overline{\delta x}^2}{\lfloor 2}\frac{d^2 t}{dx^2} + \cdots\right]^k.$$

This formula gives $d^n y/dx^n$ in terms of $d^k y/dt^k$ and $d^k t/dx^k$, which is not quite the formula asked for. Compare Greenhill's "Differential and Integral Calculus," p. 180.

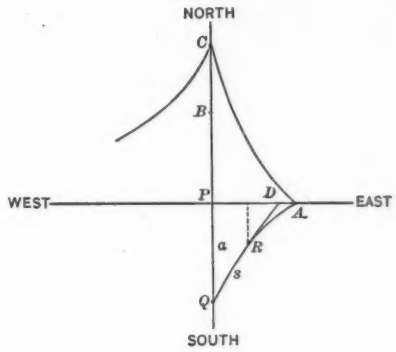
352. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

At point P there are n foxes. At Q, a rods south of P, there is a dog. The dog and the foxes are freed at the same instant and run at uniform speeds. Some of the foxes run east, some north, some west and some south. The dog runs first towards the foxes that ran east and always points toward them. He captures one of them and then instantly pursues the pack that ran north. In like manner, when he has captured one of them, he pursues those that ran west, then those that ran south, and then begins over again by pursuing the ones running east. If r is the ratio of the dog's speed to that of a fox, what is the total length of the n curves of pursuit.

(Generalization of a problem published in 1859 in the *Mathematical Monthly*.)

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

Let S be the position of a fox at any time after the chase began and R the corresponding position of the dog. Taking PA for the x-axis and PQ for the y-axis we have



$$QR = r \cdot PD, \quad \text{i. e.,} \qquad s = r\left(x - \frac{y}{\frac{dy}{dx}}\right) \qquad \text{or} \qquad \frac{ds}{dx} = r\left[1 - \frac{\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^2}\right].$$

Hence,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = ry\frac{d^2y}{dx^2} \Big/ \left(\frac{dy}{dx}\right)^2.$$

Let $\frac{dy}{dx} = p$; then $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$.

Hence, $p\sqrt{1+p^2} = ry\frac{dp}{dy}$ and therefore $\frac{dy}{y} = \frac{rdp}{p\sqrt{1+p^2}}$.

Integrating,

$$\log cy = -r\log\frac{[1+\sqrt{1+p^2}]}{p}.$$

Hence,

$$cy^{1/r} = \frac{p}{1+\sqrt{1+p^2}} \quad \text{or} \quad c^2y^{2/r}(1+p^2) = p^2 - 2cpy^{1/r} + c^2y^{2/r}.$$

Hence,

$$\frac{dy}{dx} = \frac{2cy^{1/r}}{1-c^2y^{2/r}}. \tag{1}$$

Whence,

$$\frac{1-c^2y^{2/r}}{y^{1/r}}dy = 2cdx.$$

Integrating,

$$\frac{y^{-1/r+1}}{1-\frac{1}{r}} - c^2\frac{y^{1/r+1}}{1+\frac{1}{r}} = 2cx + c'. \tag{2}$$

When

$$x = 0, \quad y = a, \quad \frac{dy}{dx} = \infty.$$

Hence, from (1),

$$c = \pm\frac{1}{a^{1/r}},$$

and, from (2),

$$c' = \frac{a^{1-1/r}}{1-\frac{1}{r}} - \frac{1}{a^{2/r}}\frac{a^{1+1/r}}{1+\frac{1}{r}}.$$

Hence, from (2),

$$\pm\frac{2x}{a^{1/r}} = \frac{y^{1-1/r}}{1-\frac{1}{r}} - \frac{1}{a^{2/r}}\frac{y^{1+1/r}}{1+\frac{1}{r}} - \frac{a^{1-1/r}}{1-\frac{1}{r}} + \frac{1}{a^{2/r}}\frac{a^{1+1/r}}{1+\frac{1}{r}}.$$

When

$$y = 0, \quad x = \pm\frac{a^{1/r}}{2}\left[-\frac{a^{1-1/r}}{1-\frac{1}{r}} + \frac{1}{a^{2/r}}\frac{a^{1+1/r}}{1+\frac{1}{r}}\right]$$

$$= \pm\frac{1}{2}\left[\frac{-a}{1-\frac{1}{r}} + \frac{a}{1+\frac{1}{r}}\right] = \mp\frac{a}{r\left(1-\frac{1}{r^2}\right)} = \mp\frac{ar}{r^2-1}.$$

Evidently the + sign must be chosen. Hence, $PA = \frac{ar}{r^2-1}$, and the length of path QA is $\frac{ar^2}{r^2-1}$.

The north-going pack is now at B, where $PB = PA$. Taking PA as y-axis and PB as x-axis, equations (1) and (2) again hold good. But this time,

when

$$x = 0, \qquad y = \frac{ar}{r^2 - 1} = a_1, \qquad \frac{dy}{dx} = -1.$$

Hence, from (1),

$$-1 = \frac{2ca_1^{1/r}}{1 - c^2 a_1^{2/r}}.$$

Whence,

$$c = \frac{1 \pm \sqrt{2}}{a_1^{1/r}}$$

and, from (2),

$$c' = \frac{a_1^{1-1/r}}{1 - \frac{1}{r}} - \frac{3 \pm 2\sqrt{2}}{a_1^{2/r}} \cdot \frac{a_1^{1/r+1}}{1 + \frac{1}{r}}.$$

Hence, from (2),

$$2(1 \pm \sqrt{2})x = \frac{y^{1-1/r}}{1 - \frac{1}{r}} a_1^{1/r} - \frac{3 \pm 2\sqrt{2}}{a_1^{1/r}} \frac{y^{1+1/r}}{1 + \frac{1}{r}} - \frac{a_1}{1 - \frac{1}{r}} + \frac{(3 \pm 2\sqrt{2})a_1}{1 + \frac{1}{r}}.$$

When

$$y = 0, \qquad x = a_1 \frac{(2 \pm 2\sqrt{2}) - (\pm 2\sqrt{2} + 4)\frac{1}{r}}{\left(1 - \frac{1}{r^2}\right) 2(1 \pm \sqrt{2})} = \frac{a_1}{1 - \frac{1}{r^2}} \left\{ 1 \mp \frac{\sqrt{2}}{r} \right\}.$$

Hence,

$$PC = \frac{a_1}{1 - \frac{1}{r^2}} \left\{ 1 + \frac{\sqrt{2}}{r} \right\}.$$

Hence, the length of the path AC is

$$\frac{a_1 r}{1 - \frac{1}{r^2}} \left\{ 1 + \frac{\sqrt{2}}{r} \right\}.$$

The length of each successive curve of pursuit will be found in the same way by multiplying the distance of the dog from P at starting, by

$$\frac{r}{1 - \frac{1}{r^2}} \left\{ 1 + \frac{\sqrt{2}}{r} \right\}.$$

Hence S, the total distance travelled by the dog, is

$$S = r \left[\frac{ar}{r^2 - 1} \left\{ 1 + \frac{r(r + \sqrt{2})}{r^2 - 1} \left(1 + \frac{r(r + \sqrt{2})}{r^2 - 1} \left((((\cdots)))\right) \right) \right\} \right]$$

$$= \frac{ar^2}{r^2 - 1} \{1 + k + k^2 + \cdots k^{n-1}\}, \qquad \text{where} \qquad k = \frac{r(r + \sqrt{2})}{r^2 - 1}.$$

$$\text{Hence, } S = \frac{ar^2}{r^2 - 1} \frac{1 - k^n}{1 - k} = \frac{ar^2}{(r^2 - 1)^n} \frac{(r^2 - 1)^n - (r^2 + \sqrt{2}r)^n}{r\sqrt{2} + 1}.$$

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

QUESTIONS.

14. In the process of solving a certain physical problem Professor H. S. Uhler of Yale University was led to the definite integral

$$\int_0^a (a^2 - x^2)x\,dx \int_{a-x}^{a+x} \frac{e^{-cy}}{y}\,dy,$$

for which he found the value

$$\frac{1}{c^2}\left[a^2 - \frac{3a}{c} + \frac{1}{c}\left(a + \frac{3}{2c}\right)(1 - e^{-2ac})\right],$$

a and c being positive constants. Professor Uhler would like to see how other persons attack the problem of evaluating this integral.

15. We are in receipt of the following communication from Mr. W. E. Heal of Washington, D. C.: "In the Proceedings of the Royal Society of Edinburgh, Vol. VII, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube.

"The writer has failed to see how this 'easy proof' follows and has been unable to find the question discussed or even mentioned in Tait's collected works. Can some reader of the MONTHLY supply the missing link or links?"

REPLY.

10. What use has been made of regular conference periods for assistance to individual students of secondary and college mathematics, and what services may they render?

I. REMARKS BY C. R. MACINNES, Princeton University.

During the last nine years the mathematical department at Princeton has tried with considerable success a scheme for assisting the students in the preparation of their work. Each evening of the week, a class-room or two is kept open and a member of the department is on hand to answer the questions that any student may ask. Attendance is purely voluntary on the part of the students and no attempt is made to keep a check on them. In hard courses like analytics, calculus or mechanics, students use this opportunity for help very freely and, in the main, very fairly. The few who regard it as a chance for free tutoring can soon be attended to, as can the man who obviously has not yet attempted the lesson himself.

An instructor has in this way a chance at the men individually when their heart is in their work, and a good deal can be done. The drawback that we have found comes from the smallness of the department; in a rush it is so much easier to work the student's problem than to point the way out of his immediate difficulty. We have as a usual thing two men on duty each evening and those of us who have done the work regard our evening "in Missouri" as about the most strenuous of the week.

It is the general opinion in our department that the plan has been a considerable help to us.

NOTES AND NEWS.

EDITED BY W. DE W. CAIRNS.

Among recent publications by the University Press of the University of California appears "A complete set of postulates for the logic of classes expressed in terms of the operation 'exception,'" by B. A. BERNSTEIN.

It sometimes happens that a teacher of mathematics desires to examine an article referred to in these columns but does not know where it may be readily obtained. The MONTHLY desires to be as helpful as possible in such matters; an inquiry directed to the chairman of this department, W. D. Cairns, Oberlin, Ohio, will ordinarily bring the desired information.

Dr. E. S. ALLEN, who has been an instructor in mathematics in Dartmouth College, has been appointed to an instructorship in Brown University.

Dr. W. F. SHENTON has been appointed instructor in mathematics at the Johns Hopkins University.

Dr. K. P. WILLIAMS, instructor in mathematics in the University of Indiana, has been advanced to an assistant professorship.

It seems certain that the war in Europe will interfere seriously with the regular receipt in this country of European scientific journals and other publications. Universities and colleges must expect many delays of this sort in the coming months.

For those who have access to the April number of the *Cornhill Magazine* a paper by Professor G. H. Bryan on "Income and Prospects of the Mathematical Specialist" will prove to be very interesting.

The *Mathematical Gazette* for January printed a report entitled "Teaching of calculus in public and secondary schools in the United Kingdom"; this is the report which was presented at Paris in April in response to question A of the International Conference on the Teaching of Mathematics.

The University of Chicago Press has published "Isolation and Measurement of the Electron" by Professor R. A. Millikan of the department of physics of the University of Chicago.

The American Book Company publishes for use this fall a wholly new edition of Raymond's "Plane Surveying for Use in the Class-room and Field." The revised text takes a form more adapted in its size and subject matter to field practice.

Volume 9 of the Supplement of the *Rendiconti del Circolo Matematico di Palermo* contains various addresses given by noted mathematicians during the celebration,

held April 14, 1914, of the thirtieth anniversary of the foundation of the "Circolo Matematico di Palermo." This society has 932 members, being the largest advanced mathematical society in the world, and it publishes a journal which Professor Landau, of Göttingen, Germany, called, in his address on the given occasion, the best mathematical journal in the world.

The April, 1914, number of the *Revista de la Sociedad Matemática Española* opens with an article by Professor G. A. Miller, entitled "Groups, nomenclature and notation."

The article by J. Sommers on elementary geometry from the standpoint of modern analysis, which appeared June 8, 1914, in the *Encyklopädie der Mathematischen Wissenschaften*, is divided into two main parts entitled "Geometry in the narrow sense" and "Spherical Trigonometry." The former is subdivided into two main divisions, entitled "Constructions" and "Applications of the theory of groups." The same number of the *Encyklopädie* contains about one hundred pages of an article by M. Zacharias entitled "Elementary geometry and elementary non-euclidean geometry treated synthetically." The earlier article, entitled "On the development of elementary geometry during the nineteenth century," was prepared by M. Simon for this *Encyklopädie*, but it was not accepted. It appeared as a supplement to the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1906.

Some of the numbers of the *Revista de la Sociedad Matemática Española* contain definitions of various mathematical terms arranged in alphabetical order. It may be observed that there is no good modern mathematical dictionary in any language. Such a dictionary in the English language is a desideratum, as it would save much time on the part of the student. The great mathematical encyclopedias which are now being published could be utilized in the preparation of such a dictionary.

Professor W. H. GARRETT, for several years the head of the department of mathematics at Baker University, has been granted a year's leave of absence and will spend the time at Harvard University.

Dr. M. O. TRIPP, head of the department of mathematics at the Muncie, Indiana, Normal School, has been elected professor of mathematics at Olivet College, Michigan.

At the convocation of the University of Chicago on August 29 thirty-four candidates were given the doctorate of philosophy. Of these, four were in the department of mathematics, namely, Meyer G. Gaba, Forbes B. Wiley, W. V. Lovitt, and Harold R. Kingston. There were also two candidates for the master's degree in mathematics.

In answer to Mr. E. B. ESCOTT's request for integral values of a, b and c which will make

$$2(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$$

a perfect square, E. Fauquembergue suggests (*L'Intermédiaire* for February) placing the last three factors equal to $2\alpha^2$, $2\beta^2$ and $2\gamma^2$ respectively. The solution is thus reduced to that of the system

$$\beta^2+\gamma^2=a^2, \qquad \gamma^2+\alpha^2=b^2, \qquad \alpha^2+\beta^2=c^2,$$

a problem solved by Euler; the simplest solution is

$$117^2+240^2=267^2, \qquad 240^2+44^2=244^2, \qquad 44^2+117^2=125^2.$$

L'Intermédiaire for January indicates in a brief note the manner in which many proofs in the theory of numbers develop by stages. In 1911 Mr. A. Cunningham reported that he had verified the fact that there is no solution in prime numbers of the congruence

$$2^{n-1}-1\equiv 0 \pmod{n^2}$$

for $n<1{,}000$. He just missed finding a solution of the congruence, for in 1913 W. Meissner proved that there is a solution for $n=1{,}093$, but that there is no other solution less than 2,000. The February number of the same journal gives the proof for $n=1{,}093$.

A paper by Professor F. CAJORI entitled "A new marking system and means of measuring mathematical abilities," which was read before the mathematics section of the Central Association of Science and Mathematics Teachers at their last meeting, is printed in full in *Science* for June 12. After maintaining that our high school texts do not differ greatly in the amount of material and difficulty of exercises and that as a consequence an experienced teacher can with due care draw up a "standard" test list of questions, Professor Cajori offers a formula for ordering students according to their ability and a method whereby to rearrange these according to an absolute standard. The formula given is

$$\text{Preliminary Mark}=\frac{M_a+rM_b+sO_a+tO_b+uD}{1+r+s+t+u},$$

where M represents grades based on memory tests (a) in daily work, (b) in examination, O represents original exercises with the same subdivisions, and D diligence shown, while r, s, t, u are the "weights" given to the last four terms in the numerator, relative to the first; the author suggests $s=1$, $r=t=u=\frac{1}{3}$. The preliminary marks are then rearranged according to Karl Pearson's theory following the Gaussian law.[1] The paper includes an auxiliary table showing "average differential abilities of pupils chosen at random" from classes of various sizes, as well as a bibliography of the principal material in this branch of statistics.

Science reports that Mr. G. N. WATSON, fellow of Trinity College, Cambridge. has been appointed to a position in pure mathematics in University College, London, to fill the vacancy created by the appointment of Dr. A. N. Whitehead

[1] *Biometrika*, Vol. 1, pp. 390–399.

to the new professorship of applied mathematics in the Imperial College of Science and Technology, London.

Relying upon the list as given in *Science* of August 21, we give the doctorates of philosophy conferred by American universities in mathematics and astronomy during the academic year 1913–14:

Johns Hopkins University—JOSIAH WESLEY GAIN, HARRY CLINTON GOSSARD, BESSIE IRVING MILLER, WALTER FRANCIS SHENTON, MABEL MINERVA YOUNG.

The University of Chicago—WILLIAM CHARLES KRATHWOHL, WILSON LEE MISER, FRANK MARION MORRISON, ELTON JAMES MOULTON, LLOYD ARTHUR HEBER WARREN.

Princeton University—RALPH DENNISON BEETLE, HAIG GALAJIKIAN, JOHN MINOR STETSON.

Harvard University—EDWARD SWITZER ALLEN, RAINARD BENTON ROBBINS.

Cornell University—ROBERT WILBUR BURGESS, ANNA HELEN TAPPAN.

University of Illinois—EDWARD AUGUST THEODORE KIRCHER, LOUIS CLARK MATHEWSON.

University of California—BENJAMIN ABRAM BERNSTEIN, DANIEL WALTER MOREHOUSE.

Clark University—JAMES ATKINS BULLARD.

Columbia University—LYMAN MORSE KELLS.

Indiana University—THOMAS EDWARD MASON.

University of Michigan—SUZAN ROSE BENEDICT.

University of Pennsylvania—STANLEY PULLIMAN SHUGERT.

The titles of the dissertations are given in *Science* for the doctorates in mathematics and all the sciences. Chemistry leads with 71, botany follows with 34, mathematics and zoölogy each have 25, physics has 23. The total in all the sciences is 241, in all other departments combined 261.

The combined departments of mathematics and mathematical astronomy at the University of Chicago conduct club meetings for reports on research biweekly during the four quarters of the year. In the summer quarter, meetings are held weekly, the alternate sessions being devoted to the discussion of pedagogical questions. The topics and leaders for the past summer quarter were as follows:

Mr. W. L. HART, "Methods of successive approximation for the solution of equations"; Professor L. E. DICKSON, "The trisection of an angle and other early Greek problems"; Professor D. R. CURTISS, "An example of Gibbs's phenomenon in double Fourier series"; Professors J. W. A. YOUNG and H. E. SLAUGHT, "Books, journals, and associations for teachers of mathematics"; Professor C. N. MOORE, "Some salient features of recent work in the theory of divergent series"; Professor G. A. BLISS, "Mathematical models of the department"; Dr. W. V. LOVITT, "A type of singular points in a transformation of three real variables"; "Open conference on the teaching of mathematics," led by Professor H. E. SLAUGHT, on topics proposed by the students through a question box.

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AN EQUATION BALANCE FOR CLASS-ROOM USE.

By E. W. PONZER, Leland Stanford Junior University.

Of equation balances, many types of which have been described, the one which, in the writer's opinion, possesses the greatest merit for demonstration work in the class-room in connection with the numerical solution of equations is that of Lalanne, presented to the French Academy of Sciences in 1840.[1] The principle on which the balance works is that of moments—a principle fundamental to many machines designed to solve equations.

Perhaps it would be in order here to briefly review the application of the principle so that the mechanical methods by which it is carried out may be clearly understood. Given the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

with real coefficients and x_1 as a root of the equation, we may consider each term to be in the form of a moment (positive or negative according to the sign of the coefficient, the root being assumed positive) where a quantity proportional to the coefficient takes the place of the weight and the moment arm is represented by one of the terms $x_1, x_1^2, x_1^3, \cdots x_1^n$. The moment is equal to zero only when x_1 is a root of the equation. A similar condition exists for any other real root of the equation. Imaginary roots would necessarily require a different treatment.

To make use of this principle in a mechanical way all that is necessary is to construct accurately the curves $y = \pm x$, $y = \pm x^2$, $y = \pm x^3$, $\cdots$ and then to use the ordinates corresponding to a real root of the equation as moment arms, together with suitable weights for the coefficients.

The figure on page 284 (showing the standard and right half of the balance) clearly illustrates the construction, which is entirely home-made. The axis on which the root is read is the x-axis. The curves up to $y = \pm x^4$ were plotted between $x = 0$ and $x = 1$ and the brass strips on which the weights are suspended were bent to shape and fastened as shown. The curves all pass through the

[1] *Comptes Rendus*, 1840, Vol. 2, pp. 859–60.